A new bijection relating q-Eulerian polynomials

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Abstract

On the set of permutations of a finite set, we construct a bijection which maps the 3-vector of statistics $(\mathrm{maj} - \mathrm{exc}, \mathrm{des}, \mathrm{exc})$ to a 3-vector $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$ associated with the q-Eulerian polynomials introduced by Shareshian and Wachs in *Chromatic quasisymmetric functions, arXiv:1405.4269(2014).*

Keywords: q-Eulerian polynomials, descents, ascents, major index, exceedances, inversions.

Notations

For all pair of integers (n, m) such that $n < m$, the set $\{n, n+1, \ldots, m\}$ is indifferently denoted by $[n, m]$,$]n-1, m]$,$[n, m+1[$ or $]n-1, m+1[$.

The set of positive integers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}_{>0}$.

For all integer $n \in \mathbb{N}_{>0}$, we denote by $[n]$ the set $[1, n]$ and by $\mathfrak{S}_n$ the set of the permutations of $[n]$. By abuse of notation, we assimilate every $\sigma \in \mathfrak{S}_n$ with the word $\sigma(1)\sigma(2)\ldots\sigma(n)$.

If a set $S = \{n_1, n_2, \ldots, n_k\}$ of integers is such that $n_1 < n_2 < \ldots < n_k$, we sometimes use the notation $S = \{n_1 < n_2 < \ldots < n_k\}$.

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1. Introduction

Let n be a positive integer and $\sigma \in \mathfrak{S}_n$. A *descent* (respectively *exceedance point*) of σ is an integer $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$). The set of descents (resp. exceedance points) of σ is denoted by $\mathrm{DES}(\sigma)$ (resp. $\mathrm{EXC}(\sigma)$) and its cardinal by $\mathrm{des}(\sigma)$ (resp. $\mathrm{exc}(\sigma)$). The integers $\sigma(i)$ with $i \in \mathrm{EXC}(\sigma)$ are called exceedance values of σ.

It is due to MacMahon [Mac15] and Riordan [Rio58] that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} = A_n(t)$$

where $A_n(t)$ is the n-th Eulerian polynomial [Eul55]. A statistic equidistributed with des or exc is said to be *Eulerian.* The statistic ides defined by $\mathrm{ides}(\sigma) = \mathrm{des}(\sigma^{-1})$ obviously is Eulerian.

The *major index* of a permutation $\sigma \in \mathfrak{S}_n$ is defined as

$$\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{DES}(\sigma)} i.$$

It is also due to MacMahon that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = \prod_{i=1}^{n} \frac{1-q^i}{1-q}.$$

A statistic equidistributed with maj is said to be *Mahonian.* Among Mahonian statistics is the statistic inv, defined by $\mathrm{inv}(\sigma) = |\mathrm{INV}(\sigma)|$ where $\mathrm{INV}(\sigma)$ is the set of *inversions* of a permutation $\sigma \in \mathfrak{S}_n$, *i.e.* the pairs of integers $(i,j) \in [n]^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

In [SW14], the authors consider analogous versions of the above statistics : let $\sigma \in \mathfrak{S}_n$, the set of *2-descents* (respectively *2-inversions*) of σ is defined as

$$\mathrm{DES}_2(\sigma) = \{i \in [n-1], \sigma(i) > \sigma(i+1) + 1\}$$

(resp.

$$\mathrm{INV}_2(\sigma) = \{1 \leq i < j \leq n, \sigma(i) = \sigma(j) + 1\})$$

and its cardinal is denoted by $\mathrm{des}_2(\sigma)$ (resp. $\mathrm{inv}_2(\sigma)$).

It is easy to see that $\mathrm{inv}_2(\sigma) = \mathrm{ides}(\sigma)$. The *2-major index* of σ is defined as

$$\mathrm{maj}_2(\sigma) = \sum_{i \in \mathrm{DES}_2(\sigma)} i.$$

By using quasisymmetric function techniques, the authors of [SW14] proved the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)}. \tag{1}$$

Similarly, by using the same quasisymmetric function method as in [SW14], the authors of [HL12] proved the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{amaj}_2(\sigma)} y^{\widetilde{\mathrm{asc}_2}(\sigma)} z^{\mathrm{ides}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{exc}(\sigma)} \tag{2}$$

where $\mathrm{asc}_2(\sigma)$ is the number of *2-ascents* of a permutation $\sigma \in \mathfrak{S}_n$, *i.e.* the elements of the set $\mathrm{ASC}_2(\sigma) = \{i \in [n-1], \sigma(i) < \sigma(i+1)+1\}$, which rises the statistic amaj_2 defined by

$$\mathrm{amaj}_2(\sigma) = \sum_{i \in \mathrm{ASC}_2(\sigma)} i,$$

and where

$$\widetilde{\mathrm{asc}_2}(\sigma) = \begin{cases} \mathrm{asc}_2(\sigma) & \text{if } \sigma(1) = 1, \\ \mathrm{asc}_2(\sigma) + 1 & \text{if } \sigma(1) \neq 1. \end{cases}$$

Definition 1.1. Let $\sigma \in \mathfrak{S}_n$. We consider the smallest 2-descent $d_2(\sigma)$ of σ such that $\sigma(i) = i$ for all $i \in [d_2(\sigma) - 1]$ (if there is no such 2-descent, we define $d_2(\sigma)$ as 0 and $\sigma(0)$ as $n+1$).

Now, let $d_2'(\sigma) > d_2(\sigma)$ be the smallest 2-descent of σ greater than $d_2(\sigma)$ (if there is no such 2-descent, we define $d_2'(\sigma)$ as n).

We define an inductive property $\mathcal{P}(d_2(\sigma))$ by :

1. $\sigma(d_2(\sigma)) < \sigma(i)$ for all $(i,j) \in \mathrm{INV}_2(\sigma)$ such that $d_2(\sigma) < i < d_2'(\sigma)$;
2. if $(d_2'(\sigma), j) \in \mathrm{INV}_2(\sigma)$ for some j, then either $\sigma(d_2(\sigma)) < \sigma(d_2'(\sigma))$, or $d_2'(\sigma)$ has the property $\mathcal{P}(d_2'(\sigma))$ (where the role of $d_2(\sigma)$ is played by $d_2'(\sigma)$ and that of $d_2'(\sigma)$ by $d_2''(\sigma)$ where $d_2''(\sigma) > d_2'(\sigma)$ is the smallest 2-descent of σ greater than $d_2'(\sigma)$, defined as n if there is no such 2-descent).

This property is well-defined because $(n,j) \notin \mathrm{INV}_2(\sigma)$ for all $j \in [n]$.

Finally, we define a statistic $\widetilde{\mathrm{des}_2}$ by :

$$\widetilde{\mathrm{des}_2}(\sigma) = \begin{cases} \mathrm{des}_2(\sigma) & \text{if the property } \mathcal{P}(d_2(\sigma)) \text{ is true,} \\ \mathrm{des}_2(\sigma)+1 & \text{otherwise.} \end{cases}$$

In the present paper, we prove the following theorem.

Theorem 1.2. *There exists a bijection $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ such that*

$$(maj_2(\sigma), \widetilde{des_2}(\sigma), inv_2(\sigma)) = (maj(\varphi(\sigma)) - exc(\varphi(\sigma)), des(\varphi(\sigma)), exc(\varphi(\sigma))).$$

As a straight corollary of Theorem 1.2, we obtain the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\widetilde{\mathrm{des}_2}(\sigma)} z^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{exc}(\sigma)} \tag{3}$$

which implies Equality (1).

The rest of this paper is organised as follows.

In Section 2, we introduce two graphical representations of a given permutation so as to highlight either the statistic $(\mathrm{maj}-\mathrm{exc}, \mathrm{des}, \mathrm{exc})$ or $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$. Practically speaking, the bijection φ of Theorem 1.2 will be defined by constructing one of the two graphical representations of $\varphi(\sigma)$ for a given permutation $\sigma \in \mathfrak{S}_n$.

We define φ in Section 3.

In Section 4, we prove that φ is bijective by constructing φ^{-1}.

2. Graphical representations

2.1. Linear graph

Let $\sigma \in \mathfrak{S}_n$. The linear graph of σ is a graph whose vertices are (from left to right) the integers $\sigma(1), \sigma(2), \ldots, \sigma(n)$ aligned in a row, where every $\sigma(k)$ (for $k \in \mathrm{DES}_2(\sigma)$) is boxed, and where an arc of circle is drawn from $\sigma(i)$ to $\sigma(j)$ for every $(i,j) \in \mathrm{INV}_2(\sigma)$.

For example, the permutation $\sigma = 34251 \in \mathfrak{S}_5$ (such that $(\mathrm{maj}_2(\sigma), \widetilde{\mathrm{des}_2}(\sigma), \mathrm{inv}_2(\sigma)) = (6,3,2)$) has the linear graph depicted in Figure 1.

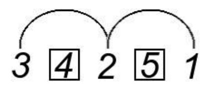

Figure 1: Linear graph of $\sigma = 34251 \in \mathfrak{S}_5$.

2.2. Planar graph

Let $\tau \in \mathfrak{S}_n$. The planar graph of τ is a graph whose vertices are the integers $1, 2, ..., n$, organized in ascending and descending slopes (the height of each vertex doesn't matter) such that the i-th vertex (from left to right) is the integer $\tau(i)$, and where every vertex $\tau(i)$ with $i \in \mathrm{EXC}(\tau)$ is encircled.

For example, the permutation $\tau = 32541 \in \mathfrak{S}_5$ (such that $(\mathrm{maj}(\tau) - \mathrm{exc}(\tau), \mathrm{des}(\tau), \mathrm{exc}(\tau)) = (6, 3, 2))$ has the planar graph depicted in Figure 2.

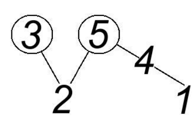

Figure 2: Planar graph of $\tau = 32541 \in \mathfrak{S}_5$.

3. Definition of the map φ of Theorem 1.2

Let $\sigma \in \mathfrak{S}_n$. We set $(r, s) = (\mathrm{des}_2(\sigma), \mathrm{inv}_2(\sigma))$, and

$$\mathrm{DES}_2(\sigma) = \left\{ d_2^k(\sigma), k \in [r] \right\},$$
$$\mathrm{INV}_2(\sigma) = \{(i_l(\sigma), j_l(\sigma)), l \in [s]\}$$

with $d_2^k(\sigma) < d_2^{k+1}(\sigma)$ for all k and $i_l(\sigma) < i_{l+1}(\sigma)$ for all l.

We intend to define $\varphi(\sigma)$ by constructing its planar graph. To do so, we first construct (in Subsection 3.1) a graph $\mathcal{G}(\sigma)$ made of n circles or dots organized in ascending or descending slopes such that two consecutive vertices are necessarily in a same descending slope if the first vertex is a circle and the second vertex is a dot. Then, in Subsection 3.2, we label the vertices of this graph with the integers $1, 2, \ldots, n$ in such a way that, if y_i is the label of the i-th vertex $v_i(\sigma)$ (from left to right) of $\mathcal{G}(\sigma)$ for all $i \in [n]$, then :

1. $y_i < y_{i+1}$ if and only if v_i and v_{i+1} are in a same ascending slope;
2. $y_i > i$ if and only if v_i is a circle.

The permutation $\tau = \varphi(\sigma)$ will then be defined as $y_1 y_2 \ldots y_n$, *i.e.* the permutation whose planar graph is the labelled graph $\mathcal{G}(\sigma)$.

With precision, we will obtain

$$\tau\,(\mathrm{EXC}(\tau)) = \{j_k(\sigma), k \in [s]\}$$

(in particular $\mathrm{exc}(\tau) = s = \mathrm{inv}_2(\sigma)$), and

$$\mathrm{DES}(\tau) = \begin{cases} \{d^k(\sigma), k \in [1, r]\} \text{ if } \widetilde{\mathrm{des}_2}(\sigma) = r, \\ \{d^k(\sigma), k \in [0, r]\} \text{ if } \widetilde{\mathrm{des}_2}(\sigma) = r + 1 \end{cases}$$

for integers $0 \leq d^0(\sigma) < d^1(\sigma) < \ldots < d^r(\sigma) \leq n$ (with $d^0(\sigma) = 0 \Leftrightarrow \widetilde{\mathrm{des}_2}(\sigma) = \mathrm{des}_2(\sigma)$) defined by

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

(with $d_2^0(\sigma) := 0$) where $(c_k(\sigma))_{k \in [0,r]}$ is a sequence defined in Subsection 3.1 such that $\sum_k c_k(\sigma) = \mathrm{inv}_2(\sigma) = \mathrm{exc}(\tau)$. Thus, we will obtain $\mathrm{des}(\tau) = \widetilde{\mathrm{des}_2}(\sigma)$ and $\mathrm{maj}(\tau) = \mathrm{maj}_2(\sigma) + \mathrm{exc}(\tau)$.

3.1. Construction of the unlabelled graph $\mathcal{G}(\sigma)$

We set $\left(d_2^0(\sigma), \sigma(d_2^0(\sigma))\right) = (0, n + 1)$ and $\left(d_2^{r+1}(\sigma), \sigma(n + 1)\right) = (n, 0)$.

For all $k \in [r]$, we define the top $t_k(\sigma)$ of the 2-descent $d_2^k(\sigma)$ as

$$t_k(\sigma) = \min\{d_2^l(\sigma), 1 \leq l \leq k, d_2^l(\sigma) = d_2^k(\sigma) - (k - l)\}, \tag{4}$$

in other words $t_k(\sigma)$ is the smallest 2-descent $d_2^l(\sigma)$ such that the 2-descents $d_2^l(\sigma), d_2^{l+1}(\sigma), \ldots, d_2^k(\sigma)$ are consecutive integers.

The following algorithm provides a sequence $(c_k^0(\sigma))_{k \in [0,r]}$ of nonnegative integers.

Algorithm 3.1. Let $I_r(\sigma) = \mathrm{INV}_2(\sigma)$. For k from $r = \mathrm{des}_2(\sigma)$ down to 0, we consider the set $S_k(\sigma)$ of sequences $(i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma))$ such that :

1. $(i_{k_p}(\sigma), j_{k_p}(\sigma)) \in I_k(\sigma)$ for all $p \in [m]$;
2. $t_k(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \ldots < i_{k_m}(\sigma)$;
3. $\sigma(i_{k_1}(\sigma)) < \sigma(i_{k_2}(\sigma)) < \ldots < \sigma(i_{k_m}(\sigma))$.

The *length* of such a sequence is defined as $l = \sum_{p=1}^{m} n_p$ where n_p is the number of *consecutive* 2-inversions whose beginning is i_{k_p}, *i.e.* the maximal number n_p of 2-inversions $(i_{k_p^1}(\sigma), j_{k_p^1}(\sigma)), (i_{k_p^2}(\sigma), j_{k_p^2}(\sigma)), \ldots, (i_{k_p^{n_p}}(\sigma), j_{k_p^{n_p}}(\sigma))$ such that $k_p^1 = k_p$ and $j_{k_p^i}(\sigma) = i_{k_p^{i+1}}(\sigma)$ for all i. If $I_k(\sigma) \neq \emptyset$, we consider the sequence $(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \ldots, i_{k_m^{max}}(\sigma)) \in I_k(\sigma)$ whose length $l^{max} = \sum_{p=1}^{m} n_p^{max}$ is maximal and whose elements $i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \ldots, i_{k_m^{max}}(\sigma)$ also are maximal (as integers). Then,

- if $I_k(\sigma) \neq \emptyset$, we set $c_k^0(\sigma) = l^{max}$ and

$$I_{k-1}(\sigma) = I_k(\sigma) \backslash \left(\cup_{p=1}^{m} \{(i_{k_i^{max}}(\sigma), j_{k_i^{max}}(\sigma)), i \in [n_p^{max}]\} \right);$$

- else we set $c_k^0(\sigma) = 0$ and $I_{k-1}(\sigma) = I_k(\sigma)$.

Example 3.2. Consider the permutation $\sigma = 549321867 \in \mathfrak{S}_9$, with $\mathrm{DES}_2(\sigma) = \{3, 7\}$ and $I_2(\sigma) = \mathrm{INV}_2(\sigma) = \{(1, 2), (2, 4), (3, 7), (4, 5), (5, 6), (7, 9)\}$. In Figure 3 are depicted the $\mathrm{des}_2(\sigma) + 1 = 3$ steps $k \in \{2, 1, 0\}$ (at each step, the 2-inversions of the maximal sequence are drawed in red then erased at the following step) :

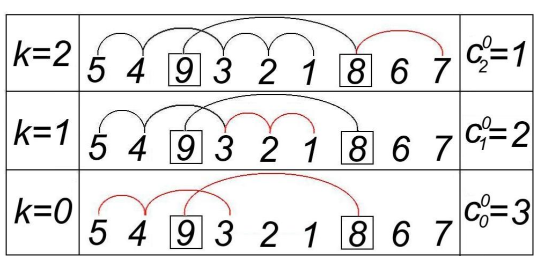

Figure 3: Computation of $(c_k^0(\sigma))_{k \in [0, \mathrm{des}_2(\sigma)]}$ for $\sigma = 549321867 \in \mathfrak{S}_9$.

- $k = 2$: there is only one legit sequence $(i_{k_1}(\sigma)) = (7)$, whose length is $l = n_1 = 1$. We set $c_2^0(\sigma) = 1$ and $I_1(\sigma) = I_2(\sigma)\backslash\{(7,9)\}$.

- $k = 1$: there are three legit sequences $(i_{k_1}(\sigma)) = (3)$ (whose length is $l = n_1 = 1$) then $(i_{k_1}(\sigma)) = (4)$ (whose length is $l = n_1 = 2$) and $(i_{k_1}(\sigma)) = (5)$ (whose length is $l = n_1 = 1$). The maximal sequence is the second one, hence we set $c_1^0(\sigma) = 2$ and $I_0(\sigma) = I_1(\sigma)\backslash\{(4,5),(5,6)\}$.

- $k = 0$: there are three legit sequences $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (1,3)$ (whose length is $l = n_1 + n_2 = 2 + 1 = 3$) then $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (2,3)$ (whose length is $l = n_1 + n_2 = 1 + 1 = 2$) and $(i_{k_1}(\sigma)) = (3)$ (whose length is $l = n_1 = 1$). The maximal sequence is the first one, hence we set $c_0^0(\sigma) = 3$ and $I_{-1}(\sigma) = I_0(\sigma)\backslash\{(1,2),(2,4),(3,7)\} = \emptyset$.

Lemma 3.3. *The sum* $\sum_k c_k^0(\sigma)$ *equals* $inv_2(\sigma)$ *(i.e.* $I_{-1}(\sigma) = \emptyset$*) and, for all* $k \in [0, r] = [0, des_2(\sigma)]$*, we have* $c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)$ *with equality only if* $c_{k+1}^0(\sigma) > 0$ *(where* $c_{r+1}^0(\sigma)$ *is defined as* 0*).*

Proof. With precision, we show by induction that, for all $k \in \{\mathrm{des}_2(\sigma), \dots, 1, 0\}$, the set $I_{k-1}(\sigma)$ contains no 2-inversion (i,j) such that $d_2^k(\sigma) < i$. For $k = 0$, it will mean $I_{-1}(\sigma) = \emptyset$ (recall that $d_2^0(\sigma)$ has been defined as 0).

$\star$ If $k = \mathrm{des}_2(\sigma) = r$, the goal is to prove that $c_r^0(\sigma) < n - d_2^r(\sigma)$. Suppose there exists a sequence $(i_{k_1}(\sigma), i_{k_2}(\sigma), \dots, i_{k_m}(\sigma))$ of length $c_r^0(\sigma) \geq n - d_2^r(\sigma)$ with $t_r(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \dots < i_{k_m}(\sigma)$. In particular, there exist $c_r^0(\sigma) \geq n - d_2^r(\sigma)$ 2-inversions (i,j) such that $d_2^r(\sigma) < j$, which forces $c_r^0(\sigma)$ to equal $n - d_2^r(\sigma)$ and every $j > d_2^r(\sigma)$ to be the arrival of a 2-inversion (i,j) such that $t_r(\sigma) \leq i$. In particular, this is true for $j = d_2^r(\sigma) + 1$, which is absurd because $\sigma(i) \geq \sigma\left(d_2^r(\sigma)\right) > \sigma\left(d_2^r(\sigma) + 1\right) + 1$ for all $i \in [t_r(\sigma), d_2^r(\sigma)]$. Therefore $c_r^0(\sigma) < n - d_2^r(\sigma)$. Also, it is easy to see that every $i > d_2^r(\sigma)$ that is the beginning of a 2-inversion (i,j) necessarily appears in the maximal sequence $\left(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \dots, i_{k_m^{max}}(\sigma)\right)$ whose length defines $c_r^0(\sigma)$, hence $(i,j) \notin I_{r-1}(\sigma)$.

$\star$ Now, suppose that $c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)$ for some $k \in [\mathrm{des}_2(\sigma)]$ with equality only if $c^0_{k+1}(\sigma) > 0$, and that no 2-inversion (i,j) with $d^k_2(\sigma) < i$ belongs to $I_{k-1}(\sigma)$.

If $t_{k-1}(\sigma) = t_k(\sigma)$ (*i.e.*, if $d^{k-1}_2(\sigma) = d^k_2(\sigma) - 1$), since $I_{k-1}(\sigma)$ does not contain any 2-inversion (i,j) with $d^k_2(\sigma) < i$, then $c^0_{k-1}(\sigma) \le 1 = d^k_2(\sigma) - d^{k-1}_2(\sigma)$. Moreover, if $c^0_{k-1}(\sigma) = 1$, then there exists a 2-inversion $(i,j) \in I_{k-1}(\sigma) \subset I_k(\sigma)$ such that $i \in [t_{k-1}(\sigma), d^k_2(\sigma)]$. Consequently (i) was a legit sequence for the computation of $c^0_k(\sigma)$ at the previous step (because $t_k(\sigma) = t_{k-1}(\sigma)$), which implies $c^0_k(\sigma)$ equals at least the length of (i). In particular $c^0_k(\sigma) > 0$.

Else, consider a sequence $(i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma))$ that fits the three conditions of Algorithm 3.1 at the step $k-1$. In particular $t_{k-1}(\sigma) \le i_{k_1}(\sigma)$. Also $i_{k_m}(\sigma) \le d^k_2(\sigma)$ by hypothesis. Since $\sigma(i_{k_p}(\sigma)) < \sigma(i_{k_{p+1}}(\sigma))$ for all p, and since $\sigma(t_{k-1}(\sigma)) > \sigma(t_{k-1}(\sigma)+1) > \ldots > \sigma\left(d^{k-1}_2(\sigma)\right) > \sigma\left(d^{k-1}_2(\sigma)+1\right)$, then only one element of the set $[t_{k-1}(\sigma), d^{k-1}_2(\sigma)+1]$ may equal $i_{k_p}(\sigma)$ for some $p \in [m]$. Thus, the length l of the sequence verifies $l \le d^k_2(\sigma) - d^{k-1}_2(\sigma)$, with equality only if $i_{k_m}(\sigma) = d^k_2(\sigma)$ (which implies $c^0_k(\sigma) > 0$ as in the previous paragraph). In particular, this is true for $l = c^0_{k-1}(\sigma)$.

Finally, as for $k = \mathrm{des}_2(\sigma)$, every $i \in [d^{k-1}_2(\sigma)+1, d^k_2(\sigma)]$ that is the beginning of a 2-inversion (i,j) necessarily appears in the maximal sequence $\left(i_{k^{max}_1}(\sigma), i_{k^{max}_2}(\sigma), \ldots, i_{k^{max}_m}(\sigma)\right)$ whose length defines $c^0_{k-1}(\sigma)$, hence $(i,j) \notin I_{k-2}(\sigma)$.

So the lemma is true by induction. $\square$

Definition 3.4. We define a graph $\mathcal{G}^0(\sigma)$ made of circles and dots organised in ascending or descending slopes, by plotting :

- for all $k \in [0,r]$, an ascending slope of $c^0_k(\sigma)$ circles such that the first circle has abscissa $d^k_2(\sigma)+1$ and the last circle has abscissa $d^k_2(\sigma)+c^0_k(\sigma)$ (if $c^0_k(\sigma) = 0$, we plot nothing). All the abscissas are distinct because

$$d^0_2(\sigma) + c_0 < d^1_2(\sigma) + c_1 < \ldots < d^r_2(\sigma) + c_r$$

  in view of Lemma 3.3;

- dots at the remaining $n-s = n-\text{inv}_2(\sigma)$ abscissas from 1 to n, in ascending and descending slopes with respect to the descents and ascents of the word $\omega(\sigma)$ defined by

$$\omega(\sigma) = \sigma(u_1(\sigma))\sigma(u_2(\sigma))\ldots\sigma(u_{n-s}(\sigma)) \tag{5}$$

where

$$\{u_1(\sigma) < u_2(\sigma) < \ldots < u_{n-s}(\sigma)\} := \mathfrak{S}_n \backslash \{i_1(\sigma) < i_2(\sigma) < \ldots < i_s(\sigma)\}.$$

Example 3.5. The permutation $\sigma_0 = 425736981 \in \mathfrak{S}_9$ (with $\text{DES}_2(\sigma_0) = \{1,4,8\}$ and $\text{INV}_2(\sigma_0) = \{(1,5),(2,9),(4,6),(7,8)\}$), which yields the sequence $(c_k^0(\sigma_0))_{k\in[0,3]} = (1,1,2,0)$ (see Figure 4 where all the 2-inversions involved in the computation of a same $c_k^0(\sigma_0)$ are drawed in a same color) and the word $\omega(\sigma_0) = 53681$, provides the unlabelled graph $\mathcal{G}^0(\sigma_0)$ depicted in Figure 5.

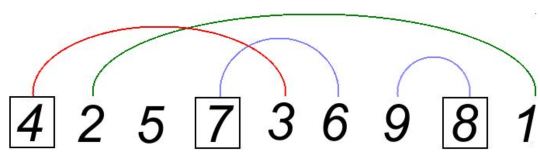

Figure 4: $(c_k^0(\sigma_0))_{k\in[0,3]} = (1,1,2,0)$.

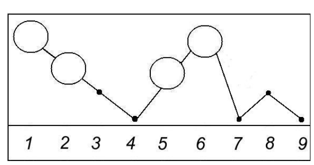

Figure 5: Graph $\mathcal{G}^0(\sigma_0)$.

The following lemma is easy.

Lemma 3.6. *For all $i \in [n]$, if the i-th vertex (from left to right) $v_i^0(\sigma)$ of $\mathcal{G}^0(\sigma)$ is a dot and if i is a descent of $\mathcal{G}^0(\sigma)$ (i.e., if $v_i^0(\sigma)$ and $v_{i+1}^0(\sigma)$ are two dots in a same descending slope) whereas $i \notin DES_2(\sigma)$, let k_i such that*

$$d_2^{k_i}(\sigma) \ + \ c_{k_i}^0(\sigma) \ < \ i < \ d_2^{k_i+1}(\sigma)$$

and let $p \in [n-s]$ such that $v_i^0(\sigma)$ is the p-th dot (from left to right) of $\mathcal{G}^0(\sigma)$. Then :

1. *$u_p(\sigma)$ is the greatest integer $u < d_2^{k_i+1}(\sigma)$ that is not the beginning of a 2-inversion of σ;*
2. *$u_{p+1}(\sigma)$ is the smallest integer $u > d_2^{k_i+1}(\sigma)$ that is not a 2-descent or the beginning of a 2-inversion of σ;*
3. *$c_k^0(\sigma) > 0$ for all k such that $d_2^{k_i+1}(\sigma) \leq d_2^k(\sigma) \leq u_{p+1}(\sigma)$.*

In particular $c_{k_i+1}^0(\sigma) > 0$.

Lemma 3.6 motivates the following definition.

Definition 3.7. For i from 1 to $n-1$, let $k_i \in [0,r]$ such that

$$d_2^{k_i}(\sigma) \ + \ c_{k_i}^0(\sigma) \ < \ i < \ d_2^{k_i+1}(\sigma).$$

If i fits the conditions of Lemma 3.6, then we define a sequence $(c_k^i(\sigma))_{k\in[0,r]}$ by

$$\begin{aligned}
c_{k_i}^i(\sigma) &= c_{k_i}^{i-1}(\sigma) + 1,\\
c_{k_i+1}^i(\sigma) &= c_{k_i+1}^{i-1}(\sigma) - 1,\\
c_k^i(\sigma) &= c_k^{i-1}(\sigma) \text{ for all } k \notin \{k_i, k_i+1\}.
\end{aligned}$$

Else, we define $(c_k^i(\sigma))_{k\in[0,r]}$ as $(c_k^{i-1}(\sigma))_{k\in[0,r]}$.

The final sequence $(c_k^n(\sigma))_{k\in[0,r]}$ is denoted by

$$(c_k(\sigma))_{k\in[0,r]}.$$

By construction, and from Lemma 3.3, the sequence $(c_k(\sigma))_{k\in[0,r]}$ has the same properties as $(c_k^0(\sigma))_{k\in[0,r]}$ detailed in Lemma 3.3.

Consequently, we may define an unlabelled graph

$$\mathcal{G}(\sigma)$$

by replacing $(c_k^0(\sigma))_{k\in[0,r]}$ with $(c_k(\sigma))_{k\in[0,r]}$ in Definition 3.4.

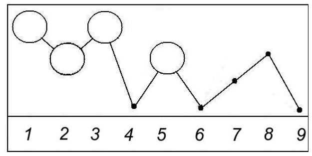

Figure 6: Graph $\mathcal{G}(\sigma_0)$.

Example 3.8. In the graph $\mathcal{G}^0(\sigma)$ depicted in Figure 5 where $\sigma_0 = 425736981 \in \mathfrak{S}_9$, we can see that the dot $v_3^0(\sigma_0)$ is a descent whereas $3 \notin \mathrm{DES}_2(\sigma_0)$, hence, from the sequence $(c_k^0(\sigma_0))_{k\in[0,3]} = (1,1,2,0)$, we compute $(c_k(\sigma_0))_{k\in[0,3]} = (1,2,1,0)$ and we obtain the graph $\mathcal{G}(\sigma_0)$ depicted in Figure 6.

Let $v_1(\sigma), v_2(\sigma), \ldots, v_n(\sigma)$ be the n vertices of $\mathcal{G}(\sigma)$ from left to right.

By construction, the descents of the unlabelled graph $\mathcal{G}(\sigma)$ (*i.e.*, the integers $i \in [n-1]$ such that $v_i(\sigma)$ and $v_{i+1}(\sigma)$ are in a same descending slope) are the integers

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

for all $k \in [0, r]$.

3.2. Labelling of the graph $\mathcal{G}(\sigma)$

3.2.1. Labelling of the circles

We intend to label the circles of $\mathcal{G}(\sigma)$ with the integers

$$j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma).$$

Algorithm 3.9. For all $i \in [n]$, if the vertex $v_i(\sigma)$ is a circle (hence $i < n$), we label it first with the set

$$[i+1, n] \cap \{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\}.$$

Afterwards, if a circle $v_i(\sigma)$ is found in a descending slope such that there exists a quantity of a circles above $v_i(\sigma)$, and in an ascending slope such that there exists a quantity of b circles above $v_i(\sigma)$, then we remove the $a+b$ greatest integers from the current label of $v_i(\sigma)$ (this set necessarily had at least $a+b+1$

elements) and the smallest integer from every of the $a+b$ labels of the $a+b$ circles above $v_i(\sigma)$ in the two related slopes. At the end of this step, if an integer $j_k(\sigma)$ appears in only one label of a circle $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with $j_k(\sigma)$.

Finally, we replace every label that is still a set by the unique integer it may contain with respect to the order of the elements in the sequence

$$(j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma))$$

(from left to right).

Example 3.10. For $\sigma_0 = 425736981$ (see Figure 4) whose graph $\mathcal{G}(\sigma_0)$ is depicted in Figure 6, we have $s = \text{inv}_2(\sigma) = 4$ and $\{j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)\} = \{5, 6, 8, 9\}$, which provides first the graph labelled by sets depicted in Figure 7. Afterwards, since the circle $v_2(\sigma_0)$ is in a descending slope with $a = 1$ circle above it (the vertex $v_1(\sigma_0)$) and in an ascending slope with also $b = 1$ circle above it (the vertex $v_3(\sigma_0)$), then we remove the $a + b = 2$ integers 8 and 9 from its label, which becomes $\{5, 6\}$, and we remove 5 from the labels of $v_1(\sigma_0)$ and $v_3(\sigma_0)$. Also, since the label of $v_2(\sigma_0)$ is the only set that contains 5, then we label $v_2(\sigma_0)$ with 5 (see Figure 8). Finally, the sequence $(j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)) = (5, 9, 6, 8)$ gives the order (from left to right) of apparition of the remaining integers $6, 8, 9$ (see Figure 9).

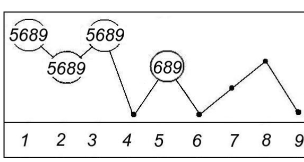

Figure 7

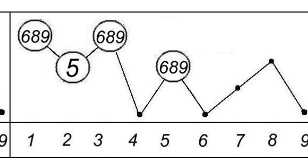

Figure 8

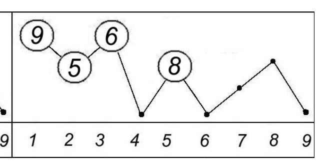

Figure 9

3.2.2. Labelling of the dots

Let

$$\{p_1(\sigma) < p_2(\sigma) < \ldots < p_{n-s}(\sigma)\} = [n] \backslash \bigsqcup_{k=0}^{r}]d_2^k(\sigma), d^k(\sigma)].$$

We intend to label the dots $\{v_{p_i(\sigma)}(\sigma), i \in [n-s]\}$ of $\mathcal{G}(\sigma)$ with the elements of

$$\{1 = e_1(\sigma) < e_2(\sigma) < \ldots < e_{n-s}(\sigma)\} = [n]\backslash\{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\}.$$

Algorithm 3.11. 1. For all $k \in [n-s]$, we label first the dot $v_{p_k(\sigma)}(\sigma)$ with the set

$$[\min(p_k(\sigma), u_k(\sigma))] \cap ([n]\backslash\{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\})$$

where $u_1(\sigma), u_2(\sigma), \ldots, u_{n-s}(\sigma)$ are the integers introduced in (5).

2. Afterwards, similarly as for the labelling of the circles, if a dot $v_i(\sigma)$ is found in a descending slope such that a dots are above $v_i(\sigma)$, and in an ascending slope such that b dots are above $v_i(\sigma)$, then we remove the $a+b$ greatest integers from the current label of $v_i(\sigma)$ and the smallest integer from every of the $a+b$ labels of the dots above $v_i(\sigma)$ in the two related slopes. At the end of this step, if an integer l appears in only one label of a dot $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with l.

3. Finally, for k from 1 to $n-s$, let

$$w_1^k(\sigma) < w_2^k(\sigma) < \ldots < w_{q_k(\sigma)}^k(\sigma) \tag{6}$$

such that

$$\{p_{w_i^k(\sigma)}(\sigma), i\} = \{p_i(\sigma),\ e_k(\sigma) \text{ appear in the label of } p_i(\sigma)\},$$

and let $i(k) \in [q_k(\sigma)]$ such that

$$\sigma\left(u_{w_{i(k)}^k(\sigma)}(\sigma)\right) = \min\{\sigma\left(u_{w_i^k(\sigma)}(\sigma)\right), i \in [q_k(\sigma)]\}.$$

Then, we replace the label of the dot $p_{w_{i(k)}^k(\sigma)}(\sigma)$ with the integer $e_k(\sigma)$ and we erase $e_k(\sigma)$ from any other label (and if an integer l appears in only one label of a dot $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with l).

Example 3.12. For $\sigma_0 = 425736981$ whose graph $\mathcal{G}(\sigma_0)$ has its circles labelled in Figure 9, the sequence $(u_1(\sigma_0), u_2(\sigma_0), u_3(\sigma_0), u_4(\sigma_0), u_5(\sigma_0)) = (3, 5, 6, 8, 9)$ provides first the graph labelled by sets depicted in Figure 10. The rest of the algorithm goes from $k = 1$ to $n - s = 9 - 4 = 5$.

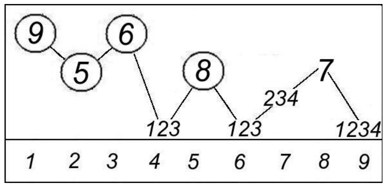

Figure 10

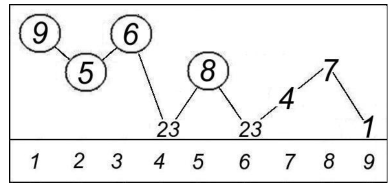

Figure 11

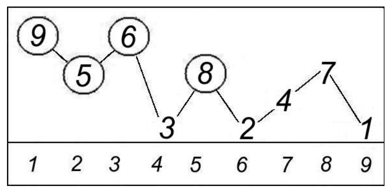

Figure 12: Labelled graph $\mathcal{G}(\sigma_0)$.

- $k = 1$: in Figure 10, the integer $e_1(\sigma_0) = 1$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$, $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ and $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$, so, from
$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0)), \sigma_0(u_5(\sigma_0)) = (5, 3, 1),$$
we label the dot $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$ with the integer $e_1(\sigma_0) = 1$ and we erase 1 from any other label, and since the integer 4 now only appears in the label of the dot $v_7(\sigma_0)$, then we label $v_7(\sigma_0)$ with 4 (see Figure 11).

- $k = 2$: in Figure 11, the integer $e_2(\sigma_0) = 2$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$ and $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ so, from
$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0))) = (5, 3),$$
we label the dot $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ with the integer $e_2(\sigma_0) = 2$ and we erase 2 from any other label, which provides the graph labelled by integers depicted in Figure 12.

- The three steps $k = 3, 4, 5$ change nothing because every dot of $\mathcal{G}(\sigma_0)$ is already labelled by an integer at the end of the previous step.

So the final version of the labelled graph $\mathcal{G}(\sigma_0)$ is the one depicted in Figure 12.

3.3. Definition of $\varphi(\sigma)$

By construction of the labelled graph $\mathcal{G}(\sigma)$, the word $y_1 y_2 \dots y_n$ (where the integer y_i is the label of the vertex $v_i(\sigma)$ for all i) obviously is a permutation of the set $[n]$, whose planar graph is $\mathcal{G}(\sigma)$.

We define $\varphi(\sigma) \in \mathfrak{S}_n$ as this permutation.

For the example $\sigma_0 = 425736981 \in \mathfrak{S}_9$ whose labelled graph $\mathcal{G}(\sigma_0)$ is depicted in Figure 12, we obtain $\varphi(\sigma_0) = 956382471 \in \mathfrak{S}_9$.

In general, by construction of $\tau = \varphi(\sigma) \in \mathfrak{S}_n$, we have

$$\tau\left(\mathrm{EXC}(\tau)\right) = \{j_k(\sigma), k \in [\mathrm{inv}_2(\sigma)]\} \tag{7}$$

and

$$\mathrm{DES}(\tau) = \begin{cases} \{d^k(\sigma), k \in [1, \mathrm{des}_2(\sigma)]\} & \text{if } c_0(\sigma) = 0 (\Leftrightarrow d^0(\sigma) = 0), \\ \{d^k(\sigma), k \in [0, \mathrm{des}_2(\sigma)]\} & \text{otherwise.} \end{cases} \tag{8}$$

Equality (7) provides

$$\mathrm{exc}(\tau) = \mathrm{inv}_2(\sigma).$$

By $d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$ for all k, Equality (8) provides

$$\mathrm{maj}(\tau) = \mathrm{maj}_2(\sigma) + \sum_{k \geq 0} c_k(\sigma),$$

and by definition of $(c_k(\sigma))_k$ and Lemma 3.3 we have $\sum_{k\geq 0} c_k(\sigma) = \sum_{k \geq 0} c_k^0(\sigma) = \mathrm{inv}_2(\sigma) = \mathrm{exc}(\tau)$ hence

$$\mathrm{maj}(\tau) - \mathrm{exc}(\tau) = \mathrm{maj}_2(\sigma).$$

Finally, it is easy to see that $\widetilde{\mathrm{des}_2}(\sigma) = \mathrm{des}_2(\sigma)$ if and only if $c_0(\sigma) = 0$, so Equality (8) also provides

$$\mathrm{des}(\tau) = \widetilde{\mathrm{des}_2}(\sigma).$$

As a conclusion, we obtain

$$(\mathrm{maj}(\tau) - \mathrm{exc}(\tau), \mathrm{des}(\tau), \mathrm{exc}(\tau)) = (\mathrm{maj}_2(\sigma), \widetilde{\mathrm{des}_2}(\sigma), \mathrm{inv}_2(\sigma))$$

as required by Theorem 1.2.

4. Construction of φ^{-1}

To end the proof of Theorem 1.2, it remains to show that $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ is surjective. Let $\tau \in \mathfrak{S}_n$. We introduce integers $r \geq 0$, $s = \text{exc}(\tau)$, and

$$0 \leq d^{0,\tau} < d^{1,\tau} < \ldots < d^{r,\tau} < n$$

such that

$$\begin{aligned} \text{DES}(\tau) &= \{d^{k,\tau}, k \in [0,r]\} \cap \mathbb{N}_{>0}, \\ d^{0,\tau} &= 0 \Leftrightarrow \tau(1) = 1. \end{aligned}$$

In particular $\text{des}(\tau) = \begin{cases} r & \text{if } \tau(1) = 1, \\ r+1 & \text{otherwise.} \end{cases}$

For all $k \in [0,r]$, we define

$$\begin{aligned} c_k^\tau &= \text{EXC}(\tau) \cap]d^{k-1,\tau}, d^{k,\tau}] \text{ (with } d^{-1,\tau} := 0), \\ d_2^{k,\tau} &= d^{k,\tau} - c_k^\tau. \end{aligned}$$

We have

$$0 = d_2^{0,\tau} < d_2^{1,\tau} < \ldots < d_2^{r,\tau} < n$$

and similarly as Formula 4, we define

$$t_k^\tau = \min\{d_2^{l,\tau}, 1 \leq l \leq k, d_2^{l,\tau} = d_2^{k,\tau} - (k-l)\} \tag{9}$$

for all $k \in [r]$.

We intend to construct a graph $\mathcal{H}(\tau)$ which is the linear graph of permutation $\sigma \in \mathfrak{S}_n$ such that $\varphi(\sigma) = \tau$.

4.1. Skeleton of the graph $\mathcal{H}(\tau)$

We consider a graph $\mathcal{H}(\tau)$ whose vertices $v_1^\tau, v_2^\tau, \ldots, v_n^\tau$ (from left to right) are n dots, aligned in a row, among which we box the $d_2^{k,\tau}$-th vertex $v_{d_2^{k,\tau}}^\tau$ for all $k \in [r]$. We also draw the end of an arc of circle above every vertex v_j^τ such that $j = \tau(i)$ for some $i \in \text{EXC}(\tau)$.

For the example $\tau_0 = 956382471 \in \mathfrak{S}_9$ (whose planar graph is depicted in Figure 12), we have $r = \mathrm{des}(\tau_0) - 1 = 3$ and

$$\begin{aligned}
(c_k^{\tau_0})_{k\in[0,3]} &= (1,2,1,0),\\
(d_2^{k,\tau_0})_{k\in[0,3]} &= (1-1, 3-2, 5-1, 8-0) = (0,1,4,8),\\
\tau_0(\mathrm{EXC}(\tau_0)) &= \{5,6,8,9\},
\end{aligned}$$

and we obtain the graph $\mathcal{H}(\tau_0)$ depicted in Figure 13.

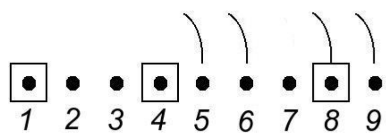

Figure 13: Incomplete graph $\mathcal{H}(\tau_0)$.

In general, by definition of $\varphi(\sigma)$ for all $\sigma \in \mathfrak{S}_n$, if $\varphi(\sigma) = \tau$, then $r = \mathrm{des}_2(\sigma)$ and $d_2^k(\sigma)$ (respectively $c_k(\sigma), d^k(\sigma), t_k(\sigma)$) equals $d_2^{k,\tau}$ (resp. $c_k^\tau, d^{k,\tau}, t_k^\tau$) for all $k \in [0,r]$ and $\{j_l(\sigma), l \in [\mathrm{inv}_2(\sigma)]\} = \tau(\mathrm{EXC}(\tau))$. Consequently, the linear graph of σ necessarily have the same skeleton as that of $\mathcal{H}(\tau)$.

The following lemma is easy.

Lemma 4.1. *If $\tau = \varphi(\sigma)$ for some $\sigma \in \mathfrak{S}_n$, then :*

1. *If $j = \tau(l)$ with $l \in EXC(\tau)$ such that $l \in]d_2^{k,\tau}, d^{k,\tau}]$, and if $(i,j) \in INV_2(\sigma)$, then $t_k^\tau \leq i$.*
2. *A pair $(i, i+1)$ cannot be a 2-inversion of σ if $i \in DES_2(\sigma)$ ($\Leftrightarrow$ if the vertex v_i^τ of $\mathcal{H}(\tau)$ is boxed).*
3. *For all pair $(l,l') \in EXC(\tau)^2$, if the labels of the two circles $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $\mathcal{G}(\sigma)$, let i and i' such that $(i,l) \in INV_2(\sigma)$ and $(i',l') \in INV_2(\sigma)$, then $i < i' \Leftrightarrow l < l'$.*

Consequently, in order to construct the linear graph of a permutation $\sigma \in \mathfrak{S}_n$ such that $\tau = \varphi(\sigma)$ from $\mathcal{H}(\tau)$, it is necessary to extend the arcs of circles of $\mathcal{H}(\tau)$ to reflect the three facts of Lemma 4.1. When a vertex is necessarily the beginning of an arc of circle, we draw the beginning of an arc of circle above it.

When there is only one vertex v_i^τ that can be the beginning of an arc of circle, we complete the latter by making it start from v_i^τ.

Example 4.2. For $\tau_0 = 956382471 \in \mathfrak{S}_9$, the graph $\mathcal{H}(\tau_0)$ becomes as depicted in Figure 14. Note that the arc of circle ending at $v_6^{\tau_0}$ cannot begin at $v_5^{\tau_0}$ because

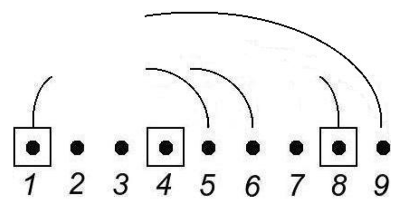

Figure 14: Incomplete graph $\mathcal{H}(\tau_0)$.

otherwise, from the third point of Lemma 4.1, and since $(6,8) = (\tau_0(l), \tau_0(l'))$ with $3 = l < l' = 5$, it would force the arc of circle ending at $v_8^{\tau_0}$ to begin at $v_{i'}^{\tau_0}$ with $6 \leq i'$, which is absurd because a permutation $\sigma \in \mathfrak{S}_9$ whose linear graph would be of the kind $\mathcal{H}(\tau_0)$ would have $c_2(\sigma) = 2 \neq 1 = c_2^{\tau_0}$. Also, still in view of the third point of Lemma 4.1, and since $\tau_0^{-1}(9) < \tau^{-1}(6)$, the arc of circle ending at $v_9^{\tau_0}$ must start before the arc of circle ending at $v_6^{\tau_0}$, hence the configuration of $\mathcal{H}(\tau_0)$ in Figure 14.

The following two facts are obvious.

Facts 4.3. If $\tau = \varphi(\sigma)$ for some $\sigma \in \mathfrak{S}_n$, then :

1. A vertex v_i^τ of $\mathcal{H}(\tau)$ is boxed if and only if $i \in \mathrm{DES}_2(\sigma)$. In that case, in particular i is a descent of σ.
2. If a pair $(i, i+1)$ is not a 2-descent of σ and if v_i^τ is not boxed, then i is an ascent of σ, *i.e.* $\sigma(i) < \sigma(i+1)$.

To reflect Facts 4.3, we draw an ascending arrow (respectively a descending arrow) between the vertices v_i^τ and v_{i+1}^τ of $\mathcal{H}(\tau)$ whenever it is known that $\sigma(i) < \sigma(i+1)$ (resp. $\sigma(i) > \sigma(i+1)$) for all $\sigma \in \mathfrak{S}_n$ such that $\varphi(\sigma) = \tau$.

For the example $\tau_0 = 956382471 \in \mathfrak{S}_9$, the graph $\mathcal{H}(\tau_0)$ becomes as depicted in Figure 15. Note that it is not known yet if there is an ascending or descending arrow between $v_7^{\tau_0}$ and $v_8^{\tau_0}$.

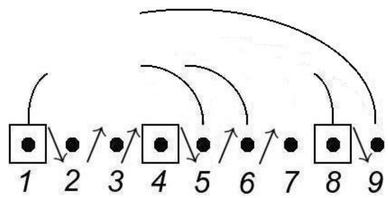

Figure 15: Incomplete graph $\mathcal{H}(\tau_0)$.

4.2. Completion and labelling of $\mathcal{H}(\tau)$

The following lemma is analogous to the third point of Lemma 4.1 for the dots instead of the circles and follows straightly from the definition of $\varphi(\sigma)$ for all $\sigma \in \mathfrak{S}_n$.

Lemma 4.4. *Let $\sigma \in \mathfrak{S}_n$ such that $\varphi(\sigma) = \tau$. For all pair $(l, l') \in ([n] \backslash EXC(\tau))^2$, if the labels of the two dots $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $\mathcal{G}(\sigma)$, let k and k' such that $l = p_k(\sigma)$ and $l' = p_{k'}(\sigma)$, then $\tau(l) < \tau(l') \Leftrightarrow \sigma(u_k(\sigma)) < \sigma(u_{k'}(\sigma))$.*

Now, the ascending and descending arrows between the vertices of $\mathcal{H}(\tau)$ introduced earlier, and Lemma 4.4, induce a partial order on the set $\{v_i^\tau, i \in [n]\}$:

Definition 4.5. We define a partial order $\succ$ on $\{v_i^\tau, i \in [n]\}$ by :

- $v_i^\tau \prec v_{i+1}^\tau$ (resp. $v_i^\tau \succ v_{i+1}^\tau$) if there exists an ascending (resp. descending) arrow between v_i^τ and v_{i+1}^τ;
- $v_i^\tau \succ v_j^\tau$ (with $i < j$) if there exists an arc of circle from v_i^τ to v_j^τ;
- if two vertices v_i^τ and v_j^τ are known to be respectively the k-th and k'-th vertices of $\mathcal{H}(\tau)$ that cannot be the beginning of a complete arc of circle, let l and l' be respectively the k-th and k'-th non-exceedance point of τ (from left to right), if (l, l') fits the conditions of Lemma 4.4, then we set $v_i^\tau \prec v_j^\tau$ (resp. $v_i^\tau \succ v_j^\tau$) if $\tau(l) < \tau(l')$ (resp. $\tau(l) > \tau(l')$).

Example 4.6. For the example $\tau_0 = 956382471$, according to the first point of Definition 4.5, the arrows of Figure 15 provide

$$v_1^{\tau_0} \succ v_2^{\tau_0} \prec v_3^{\tau_0} \prec v_4^{\tau_0} \succ v_5^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0}$$

and

$$v_8^{\tau_0} \succ v_9^{\tau_0}.$$

Definition 4.7. A vertex v_i^τ of $\mathcal{H}(\tau)$ is said to be *minimal* on a subset $S \subset [n]$ if $v_i^\tau \not\succ v_j^\tau$ for all $j \in S$.

Let

$$1 = e_1^\tau < e_2^\tau < \ldots < e_{n-s}^\tau$$

be the non-exceedance values of τ (*i.e.*, the labels of the dots of the planar graph of τ).

Algorithm 4.8. Let $S = [n]$ and $l = 1$. While the vertices $\{v_i^\tau, i \in [n]\}$ have not all been labelled with the elements of $[n]$, apply the following algorithm.

1. If there exists a unique minimal vertex v_i^τ of τ on S, we label it with l, then we set $l := l + 1$ and $S := S \backslash \{v_i^\tau\}$. Afterwards,
   (a) If v_i^τ is the ending of an arc of circle starting from a vertex v_j^τ, then we label v_j^τ with the integer l and we set $l := l + 1$ and $S := S \backslash \{v_j^\tau\}$.
   (b) If v_i^τ is the arrival of an incomplete arc of circle (in particular $i = \tau(l)$ for some $l \in \mathrm{EXC}(\tau)$), we intend to complete the arc by making it start from a vertex v_j^τ for some integer $j \in [t_k^\tau, j[$ (where $l \in]d_2^{k,\tau}, d^{k,\tau}]$) in view of the first point of Lemma 4.1. We choose v_j^τ as the rightest minimal vertex on $[t_k^\tau, j[\cap S$ from which it may start in view of the third point of Lemma 4.1, and we label this vertex v_j^τ with the integer l. Then we set $l := l + 1$ and $S := S \backslash \{v_j^\tau\}$.

   Now, if there exists an arc of circle from v_j^τ (for some j) to v_i^τ, we apply steps (a),(b) and (c) to the vertex v_j^τ in place of v_i^τ.
2. Otherwise, let $k \geq 0$ be the number of vertices v_i^τ that have already been labelled and that are not the beginning of arcs of circles. Let

$$l_1 < l_2 < \ldots < l_q$$

   be the integers $l \in [n]$ such that $l \geq \tau(l) \geq e_{k+1}^\tau$ and such that we can exchange the labels of dots $\tau(l)$ and e_{k+1}^τ in the planar graph of τ without modifying the skeleton of the graph. It is easy to see that q is precisely the number of minimal vertices of τ on S. Let $l_{i_{k+1}} = \tau^{-1}(e^{\tau}_{k+1})$ and let v^{τ}_j be the i_{k+1}-th minimal vertex (from left to right) on S. We label v^{τ}_j with l, then we set $l := l + 1$ and $S := S \backslash \{v^{\tau}_j\}$, and we apply steps 1.(a), (b) and (c) to v^{τ}_j instead of v^{τ}_i.

By construction, the labelled graph $\mathcal{H}(\tau)$ is the linear graph of a permutation $\sigma \in \mathfrak{S}_n$ such that

$$\mathrm{DES}_2(\sigma) = \{d_2^{k,\tau}, k \in [r]\}$$

and

$$\{j_l(\sigma), l \in [\mathrm{inv}_2(\sigma)]\} = \tau(\mathrm{EXC}(\tau)).$$

Example 4.9. Consider $\tau_0 = 956382471 \in \mathfrak{S}_9$ whose unlabelled and incomplete graph $\mathcal{H}(\tau_0)$ is depicted in Figure 15.

- As stated in Example 4.6, the minimal vertices of τ_0 on $S = [9]$ are $(v_2^{\tau_0}, v_5^{\tau_0}, v_9^{\tau_0})$. Following step 2 of Algorithm 4.8, $k = 0$ and the integers $l \in [9]$ such that $\tau_0(l) \geq e^{\tau_0}_{k+1} = 1$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 1 in the planar graph of τ_0 (see Figure 12) are $(l_1, l_2, l_3) = (4, 6, 9)$. By $\tau_0^{-1}(1) = 9 = l_3$, we label the third minimal vertex on $[9]$, *i.e.* the vertex $v_9^{\tau_0}$, with the integer $l = 1$.

  Afterwards, following step 1.(b), since $v_9^{\tau_0}$ is the arrival of an incomplete arc of circle starting from a vertex $v_j^{\tau_0}$ with $1 = t_1^{\tau_0} \leq j$, and with $j < 5$ because that arc of circle must begin before the arc of circle ending at $v_6^{\tau_0}$ in view of Fact 3 of Lemma 4.1, we complete that arc of circle by making it start from the unique minimal vertex $v_j^{\tau_0}$ on $[1, 5[$, *i.e.* $j = 2$, and we label $v_2^{\tau_0}$ with the integer $l = 2$ (see Figure 16). Note that as from now we know that the arc of circle ending at $v_5^{\tau_0}$ necessarily begins at $v_1^{\tau_0}$, because otherwise $v_1^{\tau_0}$, being the beginning of an arc of circle, would be the beginning of the arc of circle ending at $v_6^{\tau_0}$, which is absurd in view of Fact 3 of Lemma 4.1 because $\tau_0^{-1}(9) < \tau^{-1}(6)$, so we complete that arc of circle by making it start from $v_1^{\tau_0}$, which has been depicted in Figure 16.

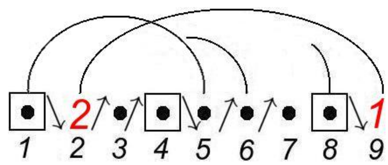

Figure 16: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = [9]\backslash\{2, 9\}$ and $l = 3$.

- From Figure 16, the minimal vertices on $S = [9]\backslash\{2, 9\}$ are $(v_3^{\tau_0}, v_5^{\tau_0})$. Following step 2 of Algorithm 4.8, $k = 1$ and the integers $l \in [9]$ such that $l \geq \tau_0(l) \geq e_{k+1}^{\tau_0} = 2$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 2 in the planar graph of τ_0 (see Figure 12) are $(l_1, l_2) = (4, 6)$. By $\tau_0^{-1}(2) = 6 = l_2$, we label the second minimal vertex on S, *i.e.* the vertex $v_5^{\tau_0}$, with the integer $l = 3$.

  Afterwards, following step 1.(a), since $v_5^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_1^{\tau_0}$, we label $v_1^{\tau_0}$ with the integer $l = 4$ (see Figure 17).

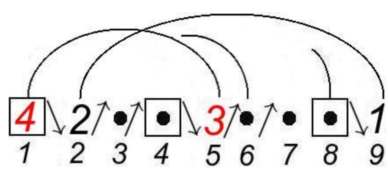

Figure 17: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = [9]\backslash\{1, 2, 5, 9\}$ and $l = 5$.

- From Figure 17, the minimal vertices on $S = \{3, 4, 6, 7, 8\}$ are $(v_3^{\tau_0}, v_6^{\tau_0})$. Following step 2 of Algorithm 4.8, $k = 2$ and the integers $l \in [9]$ such that $l \geq \tau_0(l) \geq e_{k+1}^{\tau_0} = 3$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 3 in the planar graph of τ_0 (see Figure 12) are $(l_1, l_2) = (4, 7)$. By $\tau_0^{-1}(3) = 4 = l_1$, we label the first minimal vertex on S, *i.e.* the vertex $v_3^{\tau_0}$, with the integer $l = 5$ (see Figure 18). Note that as from now we know that the arc of circle ending at $v_6^{\tau_0}$ necessarily begins at $v_4^{\tau_0}$

since it it is the only vertex left it may start from. Consequently, the arc of circle ending at $v_8^{\tau_0}$ necessarily starts from $v_7^{\tau_0}$ (otherwise it would start from $v_6^{\tau_0}$, which is prevented by Definition 4.5 because we cannot have $v_8^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0} \prec v_8^{\tau_0}$). The two latter remarks are taken into account in Figure 18.

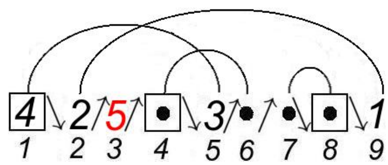

Figure 18: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = \{4, 6, 7, 8\}$ and $l = 6$.

- From Figure 18, there is only one minimal vertex on $S = \{4, 6, 7, 8\}$, *i.e.* the vertex $v_6^{\tau_0}$. Following step 1 of Algorithm 4.8, we label $v_6^{\tau_0}$ with $l = 6$.

  Afterwards, following step 1.(a), since $v_6^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_4^{\tau_0}$, we label $v_4^{\tau_0}$ with the integer $l = 7$ (see Figure 19).

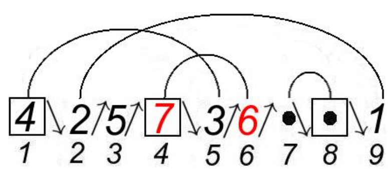

Figure 19: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

We now have $S = \{7, 8\}$ and $l = 8$.

- From Figure 19, there is only one minimal vertex on $S = \{7, 8\}$, *i.e.* the vertex $v_6^{\tau_0}$. Following step 1 of Algorithm 4.8, we label $v_6^{\tau_0}$ with $l = 8$.

  Afterwards, following step 1.(a), since $v_8^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_7^{\tau_0}$, we label $v_7^{\tau_0}$ with the integer $l = 9$ (see Figure 20).

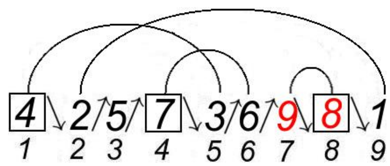

Figure 20: Labelled graph $\mathcal{H}(\tau_0)$.

As a conclusion, the graph $\mathcal{H}(\tau_0)$ is the linear graph of the permutation $\sigma_0 = 425736981 \in \mathfrak{S}_9$, which is mapped to τ_0 by φ.

Proposition 4.10. *We have $\varphi(\sigma) = \tau$, hence φ is bijective.*

Proof. By construction , for all $k \in [0, \mathrm{des}_2(\sigma)] = [0, r]$,

$$\begin{aligned} d_2^k(\sigma) &= d^{k,\tau} - c_k^\tau, \\ c_k(\sigma) &= c_k^\tau, \\ d^k(\sigma) &= d_2^k(\sigma) + c_k(\sigma) = d_2^{k,\tau} + c_k^\tau = d^{k,\tau}, \end{aligned}$$

so $\mathcal{G}(\sigma)$ has the same skeleton as the planar graph of τ, *i.e.* $\mathrm{DES}(\varphi(\sigma)) = \mathrm{DES}(\tau)$ and $\mathrm{EXC}(\varphi(\sigma)) = \mathrm{EXC}(\tau)$.

The labels of the circles of $\mathcal{G}(\sigma)$ are the elements of

$$\{j_l(\sigma), l \in [s]\} = \tau(\mathrm{EXC}(\tau)),$$

and by construction of σ, every pair $(l, l') \in \mathrm{EXC}(\tau)^2$ such that we can exchange the labels $\tau(l)$ and $\tau(l')$ in the planar graph of τ is such that

$$i < i' \Leftrightarrow l < l'$$

where $(i, \tau(l))$ and $(i, \tau(l'))$ are the two corresponding 2-inversions of σ. Consequently, by definition of $\varphi(\sigma)$, the labels of the circles of $\mathcal{G}(\sigma)$ appear in the same order as in the planar graph of τ (*i.e.* $\varphi(\sigma)(i) = \tau(i)$ for all $i \in \mathrm{EXC}(\varphi(\sigma)) = \mathrm{EXC}(\tau)$).

As a consequence, the dots of $\mathcal{G}(\sigma)$ and the planar graph of τ are labelled by the elements

$$1 = e_1(\sigma) = e_1^\tau < e_2(\sigma) = e_2^\tau < \ldots < e_{n-s}(\sigma) = e_{n-s}^\tau.$$

As for the labels of the circles, to show that the above integers appear in the same order among the labels of $\mathcal{G}(\sigma)$ and the planar graph of τ, it suffices to prove that

$$\varphi(\sigma)^{-1}(e_i^\tau) < \varphi(\sigma)^{-1}(e_j^\tau) \Leftrightarrow \tau^{-1}(e_i^\tau) < \tau^{-1}(e_j^\tau)$$

for all pair (i, j) such that we can exchange the labels e_i^τ and e_j^τ in the planar graph of τ (hence in $\mathcal{G}(\sigma)$ since the two graphs have the same skeleton). This is guaranteed by Definition 4.5 because the vertices v_i^τ that are not the beginning of an arc of circle correspond with the labels of the dots of the planar graph of τ.

As a conclusion, the planar graph of τ is in fact $\mathcal{G}(\sigma)$, *i.e.* $\tau = \varphi(\sigma)$. $\square$

5. Open problem

In view of Formula (2) and Theorem 1.2, it is natural to look for a bijection $\mathfrak{S}_n \to \mathfrak{S}_n$ that maps $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$ to $(\mathrm{amaj}_2, \widetilde{\mathrm{asc}_2}, \mathrm{ides})$.

Recall that $\mathrm{ides} = \mathrm{des}_2$ and that for a permutation $\sigma \in \mathfrak{S}_n$, the equality $\widetilde{\mathrm{des}_2}(\sigma) = \mathrm{des}_2(\sigma)$ is equivalent to $\varphi(\sigma)(1) = 1$, which is similar to the equivalence $\widetilde{\mathrm{asc}_2}(\tau) = \mathrm{asc}_2(\tau) \Leftrightarrow \tau(1) = 1$ for all $\tau \in \mathfrak{S}_n$.

Note that if $\mathrm{DES}_2(\sigma) = \bigsqcup_{p=1}^{r} [i_p, j_p]$ with $j_p + 1 < i_{p+1}$ for all p, the permutation $\pi = \rho_1 \circ \rho_2 \circ \ldots \circ \rho_r \circ \sigma$, where ρ_p is the $(j_p - i_p + 2)$-cycle

$$\begin{pmatrix} i_p & i_p + 1 & i_p + 2 & \ldots & j_p & j_p + 1 \\ \sigma(j_p + 1) & \sigma(j_p) & \sigma(j_p - 1) & \ldots & \sigma(i_p + 1) & \sigma(i_p) \end{pmatrix}$$

for all p, is such that $\mathrm{DES}_2(\sigma) \subset \mathrm{ASC}_2(\pi)$ and $\mathrm{INV}_2(\sigma) = \mathrm{INV}_2(\pi)$. One can try to get rid of the eventual unwanted 2-ascents $i \in \mathrm{ASC}_2(\pi) \backslash \mathrm{DES}_2(\sigma)$ by composing π with adequate permutations.

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